Outline

• Course information
• Motivation
• Outline of the course
• What is linear algebra?
• Chapter 1. Systems of Linear Equations
  – 1.1 Solving Linear Systems
  – 1.2 Vectors and Matrices
Course information

- Instructor: Professor Gwoboa Horng
- Textbook

*Linear Algebra : Theory and Applications*, Ward Cheney & David Kincaid,
Jones and Bartlett 1st ed. 2009 (760 pages) / 2nd ed. 2012 (648 pages)

http://www.ma.utexas.edu/CNA/LA/index.html

Errata List: http://www.ma.utexas.edu/CNA/LA/errata.html
http://www.ma.utexas.edu/CNA/LA2/errata.html
Course information

• Course web page
  http://ailab.cs.nchu.edu.tw/course/LA/100/schedule.htm

Password: ailab
Course information

• Grading (Tentative)

Homework/Quiz  20%

(You may collaborate when solving the homework. However, when writing up the solutions you must do so on your own. *Handwritten only.*)

Midterm exam  30%

Final exam  30%

Class participation  20%
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• Chapter 1. Systems of Linear Equations
  1.1 Solving Linear Systems
Purposes of the course

• To make the students become familiar with the basic concepts of linear algebra.
  – Understand matrices, vector spaces, linear transformations.

• To enhance the students' ability to reason mathematically.
  – Understand proofs, abstract notions.

• To make the students aware of the crucial importance of linear algebra to many fields in engineering, statistics and computer science.
  – Nonlinear mathematics are hard.
  – Linearizations are good approximations.
Some application areas

• Computer network
  – Network flow
• Computer graphics
  – Transformations of the plane
• Coding theory
  – Error-correcting codes
• Cryptography
  – Hill cipher
• More applications
  – http://aix1.uottawa.ca/~jkhoury/app.htm
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Outline of the course (1/2)

• Chapter 1. Systems of Linear Equations
  1.1 Solving Linear Systems
  1.2 Vectors and Matrices
  1.3 Homogeneous Linear Systems
• Chapter 2. Vector Spaces and Transformations
  2.1 Euclidean Vector Spaces
  2.2 Line, Planes, and More
  2.3 Linear Transformations
  2.4 General Vector Spaces
• Chapter 3. Matrix Operations
  3.1 Matrices
  3.2 Matrix Inverses
• Chapter 4. Determinants
  4.1 Determinants: Introduction
  4.2 Determinants: Properties and Applications
Outline of the course (2/2)

• Chapter 5. Vector Subspaces  
  5.1 Column, Row, and Null Spaces  
  5.2 Bases and Dimension  
  5.3 Coordinate Systems  

• Chapter 6. Eigensystems  
  6.1 Eigenvalues and Eigenvectors  

• Chapter 7. Inner Product Vector Spaces  
  7.1 Inner Product Spaces  
  7.2 Orthogonality  

• Chapter 8. Additional Topics  
  8.1 Hermitian Matrices and Spectral Theorem  
  8.2 Matrix Factorizations and Block Matrices  
  8.3 Iterative Methods
Outline

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  1.1 Solving Linear Systems
An algebraic structure consists of one or more sets, closed under one or more operations, satisfying some axioms.

- Group-like structures
  - \( \mathbb{Z} \) under addition (+) is an abelian group.
- Field
  - Each of \( \mathbb{Q} \), \( \mathbb{R} \), and \( \mathbb{C} \), under addition and multiplication, is a field.
What is “Linear” & “Algebra”?

• Consider a **line through the origin**
  – A directed arrow from the origin (\(\mathbf{v}\)) on the line, when scaled by a constant (\(c\)) remains on the line.
  – Two directed arrows (\(\mathbf{u}\) and \(\mathbf{v}\)) on the line can be “added” to create a longer directed arrow (\(\mathbf{u} + \mathbf{v}\)) in the same line.

• This is nothing but **arithmetic with symbols**!
  – “**Algebra**”: generalization and extension of arithmetic.
  – “**Linear**” operations: addition and scaling.

Continued in the next slide.
What is “Linear” & “Algebra”?

• **Abstract and Generalize!**
  - “Line” $\leftrightarrow$ **vector space** having N dimensions
  - “Point” $\leftrightarrow$ **vector** with N components in each of the N dimensions (**basis** vectors).
    • Vectors have: “Length” and “Direction”.
    • Basis vectors: “span” or define the space & its dimensionality.
  - Linear function transforming vectors $\leftrightarrow$ **matrix**.
    • The function acts on each vector component and scales it
    • Add up the resulting scaled components to get a new vector!
    • In general: $f(cu + dv) = cf(u) + df(v)$
What is linear algebra?

• A branch of mathematics that studies vectors, matrices, vector spaces, and systems of linear equations (p. 1)
• Vectors and matrices can produce systems of linear equations
• Systems of linear equations can often model an applied problem from the real world
Outline

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• What is linear algebra?
• **Chapter 1.** Systems of Linear Equations
  – 1.1 Solving Linear Systems
  – 1.2 Vectors and Matrices
Some advise

• Take notes
• Read in advance
• Do exercises
• Make use of the web resources
Linear equations

• Example $7x - 3y = 21$
  A single linear equation containing two variables

• The word linear derives from the word line.
Linear equations

• The **point-slope form** of a line is
  \[ y = mx + b \]
  where \( m \) is the slope and \( b \) is the intercept on the \( y \)-axis.

• Example 1: \( 7x - 3y = 21 \rightarrow y = \frac{7}{3} x - 7 \)

• For two points \((x_0,y_0)\) and \((x_1,y_1)\), the **two-point form** of the line through these points is
  \[ y - y_0 = m (x-x_0) \]
  where
  \[ m = \frac{y_1-y_0}{x_1-x_0} \]
Linear equations

• **Extend**
  
  “a single linear equation containing two variables”

  to

  “a system of $m$ linear equations containing $n$ variables (unknowns)”
Linear equations

- A linear equation in the $n$ variables

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

- $a_1, a_2, \ldots, a_n$ and $b$ are real constants
- $x_1, x_2, \ldots, x_n$ are called variables and sometimes called unknowns which do not involve any products or roots of variables.

- That is, all variables occur only to the first power and do not appear as arguments for logarithmic, trigonometric, or exponential functions.
Linear equations

- **Example:** \( x + 3y = \sin 7 \), \( y = \frac{1}{2}x + 8z + \sqrt{7} \), \( x_1 - 9x_2 - 3x_3 + x_4 = 34 \)

- **Example:** 
  \( x + 3\sqrt{y} = 5 \), \( 3x + 2y - z + xz = 4 \), and \( y = \sin x \)

**Notation:**
- 2 variables: \( x, y \)
- 3 variables: \( x, y, z \)
- \( n \) variables: \( x_1, x_2, \ldots, x_n \)
Solving Systems of Linear Equations

- A **solution** of a linear equation is a sequence of $n$ numbers $s_1, s_2, ..., s_n$ such that the equation is satisfied. The set of all solutions of the equation is called its **solution set**.

**Example:**

- Find the solution to $4x - 3y = 1$
- **Solution:**
  
  we assign an arbitrary value to $x$ and solve for $y$, or choose an arbitrary value for $y$ and solve for $x$.

  $$x = t, \quad y = \frac{4t - 1}{3} \quad \text{or} \quad x = \frac{3t + 1}{4}t, \quad y = t$$

  solution set = $$\{(t, \frac{4t - 1}{3}) \mid t \in \mathbb{R}\} \quad \text{or} \quad \{(\frac{3t + 1}{4}, t) \mid t \in \mathbb{R}\}$$

  – arbitrary number $t$ is called **parameter**.
Solving Systems of Linear Equations

- **system of linear equations (linear systems)**

  A system of \( m \) linear equations in \( n \) unknowns

  \[
  a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
  a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
  \vdots \\
  a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m
  \]

  A sequence of numbers \( s_1, s_2, \ldots, s_n \) is called a **solution** of the system if

  \[
  a_{11}s_1 + a_{12}s_2 + \ldots + a_{1n}s_n = b_1 \\
  a_{21}s_1 + a_{22}s_2 + \ldots + a_{2n}s_n = b_2 \\
  \vdots \\
  a_{m1}s_1 + a_{m2}s_2 + \ldots + a_{mn}s_n = b_m
  \]
Solving Systems of Linear Equations

Definition
A system of equations is **consistent** if it has **at least one** solution, and **inconsistent** if it has no solution.

- Every system of linear equations has either no solutions, exactly one solution, or infinitely many solutions.

Figure 1.2 Different cases of two lines in $\mathbb{R}^2$. 

(a) Unique Solution  
(b) No Solution  
(c) Infinitely Many Solutions
Solving Systems of Linear Equations

• to solve a linear system

Example:

\[
\begin{align*}
x + y + 2z &= 9 \\
2x + 4y - 3z &= 1 \\
3x + 6y - 5z &= 0
\end{align*}
\]

\[
\begin{bmatrix}
1 & 1 & 2 & 9 \\
2 & 4 & -3 & 1 \\
3 & 6 & -5 & 0
\end{bmatrix}
\]

add -2 times the first equation to the second

\[
\begin{align*}
x + y + 2z &= 9 \\
2y - 7z &= -17 \\
3x + 6y - 5z &= 0
\end{align*}
\]

\[
\begin{bmatrix}
1 & 1 & 2 & 9 \\
0 & 2 & -7 & -17 \\
3 & 6 & -5 & 0
\end{bmatrix}
\]

add -2 times the first row to the second

Continued in the next slide.
Solving Systems of Linear Equations

\[
\begin{align*}
\begin{bmatrix}
1 & 1 & 2 & \mid & 9 \\
0 & 2 & -7 & \mid & -17 \\
0 & 3 & -11 & \mid & -27 \\
\end{bmatrix}
\end{align*}
\]

Continued in the next slide.
Solving Systems of Linear Equations

\begin{align*}
\begin{bmatrix}
1 & 1 & 2 & 9 \\
0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\
0 & 0 & -\frac{1}{2} & -\frac{3}{2}
\end{bmatrix}
\end{align*}

- Multiple the third equation by -2
- Add -1 times the second equation to the first

\begin{align*}
\begin{bmatrix}
1 & 1 & 2 & 9 \\
0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\
0 & 0 & 1 & 3
\end{bmatrix}
\end{align*}

Continued in the next slide.
Solving Systems of Linear Equations

\[
\begin{align*}
x & + \frac{11}{2}z = \frac{35}{2} \\
y - \frac{7}{2}z &= -\frac{17}{2} \\
z &= 3
\end{align*}
\]

add \(-11/2\) times the third equation to the third and 
7/2 times the third equation to the second

\[
\begin{bmatrix}
1 & 0 & \frac{11}{2} \\
0 & 1 & -\frac{7}{2} \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\frac{35}{2} \\
-\frac{17}{2} \\
3
\end{bmatrix}
\]

add \(-11/2\) times the third equation to the third and 
7/2 times the third equation to the second

\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

solution
Solving Systems of Linear Equations

- **Matrix form of a Linear System**

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
  \vdots & \quad \vdots \\
  a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{align*}
\]

\[
\begin{bmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \ldots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix} =
\begin{bmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_m
\end{bmatrix}
\]

- **Augmented matrix**

\[
\begin{bmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} & | & b_1 \\
  a_{21} & a_{22} & \ldots & a_{2n} & | & b_2 \\
  \vdots & \vdots & \ddots & \vdots & | & \vdots \\
  a_{m1} & a_{m2} & \ldots & a_{mn} & | & b_m
\end{bmatrix}
\]

\[Ax = b\]

- **Ax = b**
  - \(A\): coefficient matrix
  - \(x\): unknown vector
  - \(b\): righthand side

Continued in the next slide.
Solving Systems of Linear Equations

\[
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
\vdots \quad \vdots \quad \vdots \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m
\]

\[
\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1n} & | & b_1 \\
a_{21} & a_{22} & \ldots & a_{2n} & | & b_2 \\
\vdots & \vdots & \ddots & \vdots & | & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn} & | & b_m
\end{bmatrix}
\]

Elementary Row Operations

reduced row-echelon form or row-echelon form
Solving Systems of Linear Equations

- **Elementary Row Operations**

  1. Multiply an equation through by an nonzero constant.

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{in} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
k a_{i1} & k a_{i2} & \cdots & k a_{in} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]

Continued in the next slide.
Solving Systems of Linear Equations

2. Add a multiple of one equation to another.

\[
\begin{bmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{i1} & a_{i2} & \ldots & a_{in} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{j1} & a_{j2} & \ldots & a_{jn} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{mn} & \ldots & a_{mn}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{i1} & a_{i2} & \ldots & a_{in} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{j1} & a_{j2} & \ldots & a_{jn} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{mn} & \ldots & a_{mn}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{i1} & a_{i2} & \ldots & a_{in} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{j1} & a_{j2} & \ldots & a_{jn} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{mn} & \ldots & a_{mn}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{i1} & a_{i2} & \ldots & a_{in} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{j1} & a_{j2} & \ldots & a_{jn} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{mn} & \ldots & a_{mn}
\end{bmatrix}
\]
Solving Systems of Linear Equations

3. Interchange two equation.

\[
\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{i1} & a_{i2} & \ldots & a_{in} \\
\vdots & \vdots & \ddots & \vdots \\
a_{j1} & a_{j2} & \ldots & a_{jn} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{mn} & \ldots & a_{mn} \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{j1} & a_{j2} & \ldots & a_{jn} \\
\vdots & \vdots & \ddots & \vdots \\
a_{i1} & a_{i2} & \ldots & a_{in} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{mn} & \ldots & a_{mn} \\
\end{bmatrix}
\]

Continued in the next slide.
Elementary Row Operations

• **Three types** of elementary row operations:
  1. (scale) Multiple a row by a nonzero factor.
  2. (replacement) Add a multiple of one row to another.
  3. (swap) Interchange a pair of rows.

• Let \( r_i \) and \( r_j \) be two rows.
  1. \( r_i \leftarrow k r_i \) (scale \( k \neq 0 \))
  2. \( r_j \leftarrow r_j + k r_i \) (\( i \neq j \), \( k \) is a scale)
  3. \( r_i \leftrightarrow r_j \)

• Note that **the third type of operation is redundant**
  (since it can be achieved by a sequence of operations of
  the first two types).
Let $r_1$ and $r_2$ be two rows.

With a succession of row operations of the two types (*replacement* and *scale*), we can execute a *swap*, i.e., an *interchange* of two rows:

\[
\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \sim \begin{bmatrix} r_1 \\ r_1 + r_2 \end{bmatrix} \sim \begin{bmatrix} -r_2 \\ r_1 + r_2 \end{bmatrix} \sim \begin{bmatrix} r_2 \\ r_1 + r_2 \end{bmatrix} \sim \begin{bmatrix} r_2 \\ r_1 \end{bmatrix}
\]

While only *two* types of row operation are needed, it is conventional to define *three* types.
Elementary Row Operations

- $A \sim B$ means that each matrix $A$ and $B$ can be obtained from the other by applying one or more elementary row operations.
- Two such matrices are **row equivalent** to each other, which is an **equivalence relation**:
  - $A \sim A$.
  - If $A \sim B$, then $B \sim A$.
  - If $A \sim B$ and $B \sim C$, then $A \sim C$.

- Outside of the augmented matrices, an arrow $\rightarrow$ indicates the **pivot row** and a number indicating the **multiplier** next to the **target row**

\[
\begin{align*}
\rightarrow & \begin{bmatrix} 3 & 2 & 4 \end{bmatrix} \sim -2 \begin{bmatrix} 3 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -6 \end{bmatrix} \\
-3 & \begin{bmatrix} 9 & 7 & 17 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 5 \end{bmatrix}
\end{align*}
\]
Elementary Row Operations

- When we transform a system by elementary row operations, we do not introduce spurious solutions or lose genuine solutions.
  - We can add equal quantities to equal quantities to obtain further equalities, and the process can be reverse.
Equivalent Linear Systems

• **Definition** (Equivalent Linear Systems)
  Two linear systems are said to be equivalent if one can be obtained from the other by a finite number of elementary row operations.

• **THEOREM**
  Two equivalent systems have the same set of solutions.
Equivalent Linear Systems

• **LEMMA**

Let \( Cx = d \) be the linear system obtained from the linear system \( Ax = b \) by *a single elementary row operation*. Then the linear systems \( Ax = b \) and \( Cx = d \) have the same set of solutions.
Equivalent Linear Systems

If the original pair of equations is

\[
\begin{align*}
E_1(x) &= 0 \\
E_2(x) &= 0
\end{align*}
\]

then we can proceed to

\[
\begin{align*}
E_1(x) &= 0 \\
\alpha E_1(x) + E_2(x) &= 0
\end{align*}
\]

where \( \alpha \) is any real number; i.e., a scalar.

- Every solution of the first pair of equations satisfies the second pair.
- Every solution of the second pair of equations satisfies the first pair.
- It is simply that one can **add equal quantities to equal quantities** to obtain further equalities, and the process can be reversed.
Solving Systems of Linear Equations

- **row-echelon form (ref)**
  
  A matrix is in **row-echelon form** if it satisfies

  1. All zero rows have been moved to the bottom.
  2. The leading nonzero element in any row is father to the right than the leading nonzero element in the row just above it.
  3. In each column containing a leading nonzero element, the entry below that leading nonzero element are 0.
Solving Systems of Linear Equations

- **reduced row-echelon form (rref)**
  
  A matrix is in reduced row-echelon form if it satisfies
  
  1. A row does not consist entirely of zeros → does not consist entirely of zeros
      then the first nonzero number in the row is a 1. We call this a **leading 1**
      or **pivot**
  
  2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
      Each row not all zeros

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Solving Systems of Linear Equations

3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.

\[
\begin{array}{ccccccccc}
0 & 0 & \ldots & 1 & x & x & x & x & x \\
0 & 0 & \ldots & 0 & 0 & 0 & 1 & x & x \\
\vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

4. Each column that contains a leading 1 has zeros everywhere else.
Example: reduced row-echelon form

\[
\begin{bmatrix}
0 & 1 & \times & 0 & 0 & \times \\
0 & 0 & 0 & 1 & 0 & \times \\
0 & 0 & 0 & 0 & 1 & \times \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

- This is an example of the RREF.
- Here the boxed entries are the pivot positions.
- The symbol \( \times \) designates either a zero or nonzero entry.
- Notice the descending staircase pattern of the pivots and zeros, and the fact that above and below each pivot the entries are 0’s.
Solving Systems of Linear Equations

Example: row-echelon form:

\[
\begin{bmatrix}
1 & 4 & -3 & 7 \\
0 & 1 & 7 & 5 \\
0 & 0 & 1 & 5
\end{bmatrix},
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 2 & 6 & 0 \\
0 & 0 & 1 & -1 & 3 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

Example: reduced row-echelon form:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 7 & 0 & 1 \\
0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

1. If a row does not consist entirely of zeros then the first nonzero number in the row is a 1.
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else.
Solving Systems of Linear Equations

Example: row-echelon form

\[
\begin{bmatrix}
1 & * & * & * \\
0 & 1 & * & * \\
0 & 0 & 1 & * \\
0 & 0 & 0 & 1 \\
\end{bmatrix} ,
\begin{bmatrix}
1 & * & * & * \\
0 & 1 & * & * \\
0 & 0 & 1 & * \\
0 & 0 & 0 & 0 \\
\end{bmatrix} ,
\begin{bmatrix}
1 & * & * & * \\
0 & 1 & * & * \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Example: reduced row-echelon form

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} ,
\begin{bmatrix}
1 & 0 & 0 & * \\
0 & 1 & 0 & * \\
0 & 0 & 1 & * \\
0 & 0 & 0 & 0 \\
\end{bmatrix} ,
\begin{bmatrix}
1 & 0 & * & * \\
0 & 1 & * & * \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

**Theorem 1** (will be proved later).

Every matrix has one and only one reduced row echelon form.
RREF vs REF

- It is obvious that a row echelon form is obtained with less work than is required for the reduced row echelon form.
- For some questions about a matrix (such as its rank), the row echelon form gives the answer more quickly.
- The reduced row echelon form of a matrix is unique, whereas a matrix may have many row echelon forms.
- One may ask for the reduced row echelon form of a matrix or a row echelon form.
Solving Systems of Linear Equations

- **Gaussian Elimination**

**Example:**

\[
\begin{align*}
- x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\
5x_3 + 10x_4 + 15x_6 &= 5 \\
2x_1 + 6x_2 + 8x_4 + 4x_5 - 18x_6 &= 6
\end{align*}
\]

\[
\begin{bmatrix}
1 & 3 & -2 & 0 & 2 & 0 & | & 0 \\
2 & 6 & -5 & -2 & 4 & -3 & | & -1 \\
0 & 0 & 5 & 10 & 0 & 15 & | & 5 \\
2 & 6 & 0 & 8 & 4 & -18 & | & 6
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 3 & -2 & 0 & 2 & 0 & | & 0 \\
0 & 0 & 1 & 2 & 0 & 3 & | & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & | & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & | & 0
\end{bmatrix}
\]

Augmented Matrix \quad \text{row-echelon form}

Continued in the next slide.
Solving Systems of Linear Equations

\[
\begin{bmatrix}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
x_1 + 3x_2 + 4x_4 + 2x_5 = 0 \\
x_3 + 2x_4 + 3x_6 = 0 \\
x_6 = \frac{1}{3}
\]

- Substituting \( x_6 = \frac{1}{3} \) into the 2nd equation

\[
x_1 = -3x_2 + 2x_3 - 2x_5 \\
x_3 = 1 - 2x_4 - 3x_6 \\
x_6 = \frac{1}{3}
\]

\( x_1, x_3, x_6 \): leading variables

\( x_2, x_4, x_5 \): free variables

Continued in the next slide.
Solving Systems of Linear Equations

- Substituting $x_3 = -2x_4$ into the 1st equation
  
  $x_1 = -3x_2 + 2x_3 - 2x_5$
  
  $x_3 = -2x_4$
  
  $x_6 = \frac{1}{3}$

- Assign free variables, the general solution is given by the formulas.

  $x_1 = -3r - 4s - 2t$,  $x_2 = r$,  $x_3 = -2s$,  $x_4 = s$,  $x_5 = t$,  $x_6 = \frac{1}{3}$
Solving Systems of Linear Equations

- **Gaussian-Jordan Elimination**

**Example:**

\[
\begin{align*}
  x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\
  2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\
  5x_3 + 10x_4 + 15x_6 &= 5 \\
  2x_1 + 6x_2 + 8x_4 + 4x_5 - 18x_6 &= 6
\end{align*}
\]

Augmented Matrix

\[
\begin{bmatrix}
  1 & 3 & -2 & 0 & 2 & 0 & | & 0 \\
  2 & 6 & -5 & -2 & 4 & -3 & | & -1 \\
  0 & 0 & 5 & 10 & 0 & 15 & | & 5 \\
  2 & 6 & 0 & 8 & 4 & -18 & | & 6 \\
\end{bmatrix}
\]  

\[
\begin{bmatrix}
  1 & 3 & 0 & 4 & 2 & 0 & | & 0 \\
  0 & 0 & 1 & 2 & 0 & 0 & | & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 & | & \frac{1}{3} \\
  0 & 0 & 0 & 0 & 0 & 0 & | & 0 \\
\end{bmatrix}
\]

Continued in the next slide.
Solving Systems of Linear Equations

\[
\begin{align*}
x_1 & + 3x_2 + 4x_4 + 2x_5 = 0 \\
x_3 + 2x_4 & = 0 \\
x_6 & = \frac{1}{3}
\end{align*}
\]

\[
\begin{align*}
x_1 & = -3x_2 - 4x_4 - 2x_5 \\
x_3 & = -2x_4 \\
x_6 & = \frac{1}{3}
\end{align*}
\]

\[
x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = \frac{1}{3}
\]
Algorithm for the reduced row echelon form (p. 26)

- Use blackboard
Important Terms

• **Pivot position:** a position of a leading entry in an echelon form of the matrix.

• **Pivot:** a nonzero number that either is used in a pivot position to create 0’s or is changed into a leading 1, which in turn is used to create 0’s.

• **Pivot column:** a column that contains a pivot position.
Algorithm for the reduced row echelon form

• Row reduce the matrix $A$ below to echelon form and locate the pivot columns of $A$.

\[
A = \begin{bmatrix}
0 & -3 & -6 & 4 & 9 \\
-1 & -2 & -1 & 3 & 1 \\
-2 & -3 & 0 & 3 & -1 \\
1 & 4 & 5 & -9 & -7 \\
\end{bmatrix}
\]

• **Step 1:** Interchange rows if necessary to place all zero rows on the bottom.
Algorithm for the reduced row echelon form

• **Step 2(a):** Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

\[
\begin{bmatrix}
0 & -3 & -6 & 4 & 9 \\
-1 & -2 & -1 & 3 & 1 \\
-2 & -3 & 0 & 3 & -1 \\
1 & 4 & 5 & -9 & -7
\end{bmatrix}
\]

Continued in the next slide.
Algorithm for the reduced row echelon form

• **Step 2(b):** Select a nonzero entry in the pivot column as a pivot. If necessary interchange rows to move this entry into the pivot position.

\[
\begin{bmatrix}
0 & -3 & -6 & 4 & 9 \\
-1 & -2 & -1 & 3 & 1 \\
-2 & -3 & 0 & 3 & -1 \\
1 & 4 & 5 & -9 & -7
\end{bmatrix}
\]

\( R_1 \leftrightarrow R_4 \)

Continued in the next slide.
Algorithm for the reduced row echelon form

• **Step 2(b):** Select a nonzero entry in the pivot column as a pivot. If necessary interchange rows to move this entry into the pivot position.

\[
\begin{bmatrix}
1 & 4 & 5 & -9 & -7 \\
-1 & -2 & -1 & 3 & 1 \\
-2 & -3 & 0 & 3 & -1 \\
0 & -3 & -6 & 4 & 9 \\
\end{bmatrix}
\]

Continued in the next slide.
Algorithm for the reduced row echelon form

• **Step 2(c):** Use elementary row operations to create zeros in all positions below the pivot.

\[
\begin{bmatrix}
1 & 4 & 5 & -9 & -7 \\
-1 & -2 & -1 & 3 & 1 \\
-2 & -3 & 0 & 3 & -1 \\
0 & -3 & -6 & 4 & 9 \\
\end{bmatrix}
\]

\[R_2 \rightarrow R_2 + R_1\]

\[R_3 \rightarrow R_3 + 2R_1\]

Continued in the next slide.
Algorithm for the reduced row echelon form

• After a few computations we get

\[
\begin{bmatrix}
1 & 4 & 5 & -9 & -7 \\
0 & 2 & 4 & -6 & -6 \\
0 & 5 & 10 & -15 & -15 \\
0 & -3 & -6 & 4 & 9
\end{bmatrix}
\]
Algorithm for the reduced row echelon form

• **Step 3:** Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1-2 to the remaining submatrix. Repeat the process until there are no more nonzero rows to modify.

\[
\begin{bmatrix}
1 & 4 & 5 & -9 & -7 \\
0 & 2 & 4 & -6 & -6 \\
0 & 5 & 10 & -15 & -15 \\
0 & -3 & -6 & 4 & 9 \\
\end{bmatrix}
\]

Possible Pivots

\[
R_3 \rightarrow R_3 - \frac{5}{2} R_2
\]

\[
R_4 \rightarrow R_4 + \frac{3}{2} R_2
\]

Continued in the next slide.
Algorithm for the reduced row echelon form

\[
\begin{bmatrix}
1 & 4 & 5 & -9 & -7 \\
0 & 2 & 4 & -6 & -6 \\
0 & 5 & 10 & -15 & -15 \\
0 & -3 & -6 & 4 & 9 \\
\end{bmatrix}
\]

\[R_3 \rightarrow R_3 - \frac{5}{2} R_2\]

\[R_4 \rightarrow R_4 + \frac{3}{2} R_2\]

Continued in the next slide.
Algorithm for the reduced row echelon form

\[
\begin{bmatrix}
1 & 4 & 5 & -9 & -7 \\
0 & 2 & 4 & -6 & -6 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -5 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[R_3 \leftrightarrow R_4\]

Pivot Columns

Continued in the next slide.
Algorithm for the reduced row echelon form

• **Step 4:** Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

\[
\begin{bmatrix}
1 & 4 & 5 & -9 & -7 \\
0 & 2 & 4 & -6 & -6 \\
0 & 0 & 0 & -5 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[R_3 \rightarrow -\frac{1}{5} R_3\]
Algorithm for the reduced row echelon form

\[
\begin{bmatrix}
1 & 4 & 5 & -9 & -7 \\
0 & 2 & 4 & -6 & -6 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[ R_1 \rightarrow R_1 + 9R_3 \]
\[ R_2 \rightarrow R_2 + 6R_3 \]

\[
\begin{bmatrix}
1 & 4 & 5 & 0 & -7 \\
0 & 2 & 4 & 0 & -6 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[ R_2 \rightarrow \frac{1}{2}R_2 \]

• **Step 5**: Repeat step 4, ending with the unique reduced row echelon form of the given matrix.

Continued in the next slide.
Algorithm for the reduced row echelon form

\[
\begin{bmatrix}
1 & 4 & 5 & 0 & -7 \\
0 & 1 & 2 & 0 & -3 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\Rightarrow R_1 \rightarrow R_1 - 4R_2
\]

\[
\begin{bmatrix}
1 & 0 & -3 & 0 & 5 \\
0 & 1 & 2 & 0 & -3 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Reduced Row Echelon Form
More examples

• TRANSFORMING MATRIX TO ROW ECHELON FORM
  – example01-2.htm

• TRANSFORMING MATRIX TO THE REDUCED ROW ECHELON FORM
  – example01-3.htm
Outline

• Course information
• Motivation
• Outline of the course
• What is linear algebra?
• Chapter 1. Systems of Linear Equations
  – 1.1 Solving Linear Systems
  – 1.2 Vectors and Matrices
Vectors and Matrices

- **vectors**

  \[
  \mathbf{x} = \begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n \\
  \end{bmatrix} = [x_1, x_2, \ldots, x_n]^T = (x_1, x_2, \ldots, x_n)
  \]

  - component \( x_i \): \( i \)th entry

  - **zero vector**: \( \mathbf{0} = (0, 0, \ldots, 0) \)

  - **vector addition**: \( \mathbf{x} = (x_1, x_2, \ldots, x_n), \mathbf{y} = (y_1, y_2, \ldots, y_n) \)

    \[
    \mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)
    \]

    (Figure 1.3)

  - **scalar multiplication**: \( a\mathbf{x} = (ax_1, ax_2, \ldots, ax_n) \) (Figure 1.4)

- **\( n \)-space** \( \mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\} \)
Vectors and Matrices

Figure 1.3 Addition of pairs of vectors in $\mathbb{R}^2$. 

(2, 4) + (4, 1) = (6, 5)

(-3, 4) + (2, 3) = (-1, 7)
Vectors and Matrices

Figure 1.4 Scalar multiples of vectors in $\mathbb{R}^2$. 

$2(1, 3) = (2, 6)$
$-(3, 2) = (-3, -2)$
Vectors and Matrices

• **linear combinations of vectors**
  A vector $w$ is a linear combination of the vectors $v_1, v_2, \ldots, v_n$ if it can be expressed in the form
  \[ w = k_1 v_1 + k_2 v_2 + \ldots + k_n v_n, \]
  where $k_i$ is a scalar.

  **Example:**
  $$\begin{bmatrix} -8 \\ 35 \\ -15 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} - 5 \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \end{bmatrix}$$

**Example 1 (p. 44/36):**
Every point in $\mathbb{R}^2$ is a linear combination of $e_1 = (1, 0)$ and $e_2 = (0, 1)$.
**Reason:** for any point $(x, y) = x(1, 0) + y(0, 1)$
Example 2 (p. 45/36):
Is every point in $\mathbb{R}^2$ a linear combination of $(5, 2)$ and $(7, 3)$.

**Solution:** for any point $b = (b_1, b_2)$, we try to solve the equation

$$x(5, 2) + y(7, 3) = (b_1, b_2)$$

$$\begin{bmatrix} 5 & 7 & b_1 \\ 2 & 3 & b_2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3b_1 - 7b_2 \\ 0 & 1 & -2b_1 + 5b_2 \end{bmatrix}$$

$$x = 3b_1 - 7b_2$$

$$y = -2b_1 + 5b_2$$
Vectors and Matrices

• **linear combinations of vectors**
  A vector \( \mathbf{w} \) is a linear combination of the vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \) if it can be expressed in the form
  \[
  \mathbf{w} = k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \ldots + k_n \mathbf{v}_n,
  \]
  where \( k_i \) is a scalar.

**Example 3 (p. 45/37):**
Is the \( \mathbf{w} = (-1, 3, 7) \) a linear combination of \( (4, 2, 7) \) and \( (3, 1, 4) \)?
• to solve the linear system \( x(4, 2, 7) + y(3, 1, 4) = (-1, 3, 7) \)
  \[
  4x + 3y = -1 \\
  2x + y = 3 \\
  7x + 4y = 7
  \]
Example (3)

Is the vector $\mathbf{w} = (-1, 3, 7)$ a linear combination of the vectors $\mathbf{u} = (4, 2, 7)$ and $\mathbf{v} = (3, 1, 4)$?

- We want to solve the vector equation

$$x \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 7 \end{bmatrix}$$

- The augmented matrix and its reduced row echelon form are

$$\begin{bmatrix} 4 & 3 & -1 \\ 2 & 1 & 3 \\ 7 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

- Yes, but the vectors that are linear combinations of $\mathbf{u}$ and $\mathbf{v}$ lie on a plane in $\mathbb{R}^3$. 

example02-1.htm
Vectors and Matrices

Now let $A$ be a $m \times n$ matrix, thought of as a collection of column vectors:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_1 \ a_2 \ \cdots \ a_n]$$

Here $a_j$ denotes the $j$-th column vector in $A$, which is

$$a_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$
Vectors and Matrices

**Definition**

The **matrix-vector product** $\mathbf{Ax}$ of an $m \times n$ matrix $\mathbf{A}$ and a column vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is defined to be

$$\mathbf{Ax} = x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \cdots + x_n \mathbf{a}_n$$

Here the **scalars** $x_j$ are the components of the column vector $\mathbf{x}$, and the **column vectors** $\mathbf{a}_j$ are the columns of $\mathbf{A}$. 
Vectors and Matrices

- By this definition, we have

\[ Ax = [a_1 \ a_2 \ \cdots \ a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{j=1}^{n} x_j a_j \]

- To recover one component of \( Ax \), write

\[ (Ax)_i = \sum_{j=1}^{n} x_j [a_j]_i = \sum_{j=1}^{n} x_j a_{ij} = \sum_{j=1}^{n} a_{ij} x_j = r_i x \]

- Thus, to get the \( i \)-th component of \( Ax \), we compute the vector product

\[ r_i x = [a_{i1} \ a_{i2} \ \cdots \ a_{in}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{j=1}^{n} a_{ij} x_j \]
Vectors and Matrices

Example (4)
Express as a **single vector** the product

\[
\begin{bmatrix}
1 & 5 \\
3 & 1 \\
2 & 4 \\
\end{bmatrix}
\begin{bmatrix}
2 \\
7 \\
\end{bmatrix}
\]

• We must carry out a calculation to do this:

\[
2 \begin{bmatrix}
1 \\
3 \\
2 \\
\end{bmatrix} + 7 \begin{bmatrix}
5 \\
1 \\
4 \\
\end{bmatrix} = \begin{bmatrix}
2 \\
6 \\
4 \\
\end{bmatrix} + \begin{bmatrix}
35 \\
7 \\
28 \\
\end{bmatrix} = \begin{bmatrix}
37 \\
13 \\
32 \\
\end{bmatrix}
\]

•
Vectors and Matrices

**Definition**
Let \( S = \{v_1, v_2, ..., v_n\} \). The set of all linear combinations, denoted as \( \text{span}(S) \), of a set of vectors is called the **span** of \( S \).

**Example:** (Example 1 and Example 2)
\[
\mathbb{R}^2 = \text{span}(\{(1, 0), (0,1)\}) = \text{span}(\{(5, 2), (7, 3)\})
\]

**Example 7 (p. 49/41):**
Is \((42, 6, 76)\) in the span of \((1, 2, 11), (3, 1, 4), \text{and} (7, -4, 3)\)?

**Solution:** The question is whether a solution exists for the equation

\[
\begin{bmatrix}
42 \\
6 \\
76
\end{bmatrix}
= a \begin{bmatrix}
1 \\
2 \\
11
\end{bmatrix} + b \begin{bmatrix}
3 \\
1 \\
4
\end{bmatrix} + c \begin{bmatrix}
7 \\
-4 \\
3
\end{bmatrix}
\]

Use program.  [example02-2.htm](example02-2.htm)
Example 6 (p. 49/40):
Give a simple description for the $\text{span} \{ (4, 2, 7), (3, 1, 4) \}$.

solution: for any $(b_1, b_2, b_3)$ try to solve for scalars $x$ and $y$ in

$$x(4, 2, 7) + y(3, 1, 4) = (b_1, b_2, b_3)$$

$$\begin{bmatrix} 4 & 3 & b_1 \\ 2 & 1 & b_2 \\ 7 & 4 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4b_2 - b_3 \\ 0 & 1 & b_1 - 2b_2 \\ 0 & 0 & b_1 + 5b_2 - 2b_3 \end{bmatrix}$$

for the consistency, we require $b_1 + 5b_2 - 2b_3 = 0$

$\text{span} \{ (4, 2, 7), (3, 1, 4) \} = \{ (b_1, b_2, b_3) \mid b_1 + 5b_2 - 2b_3 = 0 \}$

By simple description, we mean a simple test that can be applied to a vector to determine whether it is or is not in the span of a given set.
Interpreting linear systems

\[ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \]
\[ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \]
\[ \vdots \]
\[ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \]

⇒ \[ x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \ldots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \]

\[ A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \]

\[ Ax = b \text{ is consistent} \]
⇒ \[ b \text{ is a linear combination of} \]
\[ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ldots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \]
Interpreting linear systems

- **Equivalent forms of** \( Ax = b \) (pp. 51/43)
  
  (1) matrix form \( Ax = b \)
  
  (2) a compact summation \( \sum_{j=1}^{n} a_{ij} x_j = b_i, 1 \leq i \leq m \)
  
  (3) linear equations in complete detail

\[
\begin{align*}
  a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &= b_1 \\
  a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &= b_2 \\
  \vdots & \quad \vdots \\
  a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n &= b_m
\end{align*}
\]

Continued in next slide.
Interpreting linear systems

(4) a matrix with vectors

\[
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_m
\end{bmatrix}
= 
\begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_m
\end{bmatrix}
\]

(5) an augmented matrix

\[
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{bmatrix}
\]

(6) a linear combination of the **columns** of A

\[
x_1 \begin{bmatrix}
a_{11} \\
a_{21} \\
\vdots \\
a_{m1}
\end{bmatrix} + x_2 \begin{bmatrix}
a_{12} \\
a_{22} \\
\vdots \\
a_{m2}
\end{bmatrix} + \cdots + x_n \begin{bmatrix}
a_{1n} \\
a_{2n} \\
\vdots \\
a_{mn}
\end{bmatrix} = \begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_m
\end{bmatrix}
\]

(7) a linear combination of the **column vectors** of A
Definition (p. 52/44)
Two matrices are row equivalent to each other if each can be obtained from the other by applying a sequence of elementary (permitted) row operations.

Example:
\[
\begin{bmatrix}
5 & -1 \\
7 & 3
\end{bmatrix}
\overset{r_2\leftarrow(-1)r_1+r_2}{\sim}
\begin{bmatrix}
5 & -1 \\
2 & 4
\end{bmatrix}
\overset{r_2\leftarrow(1/2)r_2}{\sim}
\begin{bmatrix}
5 & -1 \\
1 & 2
\end{bmatrix}
\overset{r_1\leftrightarrow r_2}{\sim}
\begin{bmatrix}
1 & 2 \\
5 & -1
\end{bmatrix}
\]
row equivalent systems

**Theorem (1)**

Let two linear systems of equations be represented by their augmented matrices.

If these two augmented matrices are row equivalent to each other, then the solutions of the two systems are identical.

- In symbols:

\[[A \mid b] \sim [B \mid c] \implies \{x : Ax = b\} = \{x : Bx = c\}\]
row equivalent systems

Example:

\[
\begin{bmatrix}
5 & -1 & 11 \\
7 & 3 & 33
\end{bmatrix} \sim
\begin{bmatrix}
5 & -1 & 11 \\
2 & 4 & 22
\end{bmatrix} \sim
\begin{bmatrix}
5 & -1 & 11 \\
1 & 2 & 11
\end{bmatrix} \sim
\begin{bmatrix}
1 & 2 & 11 \\
5 & -1 & 11
\end{bmatrix} \sim \cdots \sim
\begin{bmatrix}
1 & 0 & 3 \\
0 & 1 & 4
\end{bmatrix}
\]

\[5x - y = 11\]
\[7x + 3y = 33\]

\[x = 3\]
\[y = 3\]

Example 8 (p. 53/45)

Use blackboard.
Example 8

Example (8)

Let

\[
A = \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
10 & 11 & 12
\end{bmatrix}, \quad b = \begin{bmatrix}
20 \\
47 \\
74 \\
101
\end{bmatrix}
\]

Find all the solutions to the equation

\[
Ax = b
\]
Example 8

We form the **augmented matrix** and undertake the row reduction:

\[
\begin{bmatrix}
1 & 2 & 3 & | & 20 \\
4 & 5 & 6 & | & 47 \\
7 & 8 & 9 & | & 74 \\
10 & 11 & 12 & | & 101 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & 2 & 3 & | & 20 \\
3 & 3 & 3 & | & 27 \\
6 & 6 & 6 & | & 54 \\
9 & 9 & 9 & | & 81 \\
\end{bmatrix} \sim \\
\begin{bmatrix}
1 & 2 & 3 & | & 20 \\
1 & 1 & 1 & | & 9 \\
1 & 1 & 1 & | & 9 \\
1 & 1 & 1 & | & 9 \\
\end{bmatrix} \sim \cdots \sim \\
\begin{bmatrix}
1 & 0 & -1 & | & -2 \\
0 & 1 & 2 & | & 11 \\
0 & 0 & 0 & | & 0 \\
0 & 0 & 0 & | & 0 \\
\end{bmatrix}
\]
Example 8

- It is more illuminating to write the solution as follows:

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
-2 \\
11 \\
0
\end{bmatrix} + s \begin{bmatrix}
1 \\
-2 \\
1
\end{bmatrix}
\]

Here \( s \) is a free parameter standing for the free variable \( x_3 \).

- It is customary to treat the variables in their natural order.

- But in the example just given, we could go counter to this custom and treat \( x_1 \) as the free variable.

- Now the general solution would be written

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
0 \\
7 \\
2
\end{bmatrix} + t \begin{bmatrix}
1 \\
-2 \\
1
\end{bmatrix}
\]

where the free parameter \( t \) is used in place of \( x_1 \).
Vectors and Matrices

The span of the set of columns in a matrix $A$ is called the **column space** of $A$ and is written $\text{Col}(A)$.

If $A$ is $m \times n$, then the span of the set of its columns is

$$\text{Col}(A) = \text{Span}\{ \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \} = \{ \mathbf{Ax} : x \in \mathbb{R}^n \}$$
**Theorem 2 (p. 56/48).**
A system of linear equations $Ax = b$ is consistent if and only if the vector $b$ is in the span of the set of columns of $A$. 
Vectors and Matrices

**Theorem 3 (p. 56/48).**
Let $A$ be an $m \times n$ matrix. The system of equations $Ax = b$ is consistent for all $b$ in $\mathbb{R}^m$ if and only if the columns of $A$ span $\mathbb{R}^m$. That is, $\text{col}(A) = \mathbb{R}^m$.

\[
\begin{align*}
 a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
 a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
 &\vdots \\
 a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{align*}
\]
Theorem 4 (p. 56/48).

Let $A$ be an $m \times n$ matrix. The system of equations $Ax = b$ is consistent for all $b$ in $\mathbb{R}^m$ if and only if each row of the coefficient matrix $A$ has a pivot position.

$$
\begin{bmatrix}
1 & * & * & \ldots & * \\
0 & 1 & * & \ldots & * \\
0 & 0 & 1 & \ldots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{bmatrix}
$$
Theorem 5 (p. 57/48).
A system of linear equations is *inconsistent* if and only if its augmented matrix has a pivot position in the last column.

\[
\begin{bmatrix}
0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & 0 \\
\end{bmatrix}
\]
Vectors and Matrices

A restatement of Theorem 5

**Theorem 6 (p. 58/50).**
A system of linear equations is consistent if and only if the reduced row echelon form of its augmented matrix does not have a pivot position in the last column.
Questions?